Model-Independent Extraction of $|V_{cb}|$ Using Dispersion Relations

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We present a method for parametrizing heavy meson semileptonic form factors using dispersion relations, and from it produce a two-parameter description of the $B \to B$ elastic form factor. We use heavy quark symmetry to relate this function to $\bar{B} \to D^* l \bar{\nu}$ form factors, and extract $|V_{cb}| = 0.037^{+0.003}_{-0.002}$ from experimental data with a least squares fit. Our method eliminates model-dependent uncertainties inherent in choosing a parametrization for the extrapolation of the differential decay rate to threshold. The method also allows a description of $\bar{B} \to D l \bar{\nu}$ form factors accurate to 1% in terms of two parameters.

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1. Introduction

A nonperturbative, model-independent description of QCD form factors is a desirable ingredient for the extraction of Cabibbo-Kobayashi-Maskawa parameters from exclusive meson decays. Progress towards this goal has been realized by the development of heavy quark symmetry[1], which relates and normalizes the $\bar{B} \to D^* l\bar{\nu}$ and $\bar{B} \to Dl\bar{\nu}$ form factors in the context of the $\frac{1}{M}$ expansion, where M is the heavy quark mass. This normalization has been used[2–4] to extract the value of the CKM parameter $|V_{cb}|$ by extrapolating the measured form factor to zero recoil, where the normalization is predicted.

This form factor extrapolation, necessary because the rate vanishes at zero recoil, introduces an uncertainty in the value of $|V_{cb}|$ due to the choice of parametrization. Estimates of this uncertainty obtained by varying parametrizations suffer the same ambiguity. This ambiguity could be eliminated if one had a nonperturbative, model-independent characterization of the form factor in terms of a small number of parameters.

In this paper we use dispersion relations to derive such a characterization and apply it towards the extraction of $|V_{cb}|$. The characterization uses two parameters that describe the form factor over the entire physical range to 1% accuracy. For computational convenience we use heavy quark symmetry in our characterization, but other than the normalization at threshold, this is an inessential ingredient which may be discarded at the cost of some extra algebra.

In Sec. 2 we describe a well-known method[5] for using QCD dispersion relations and analyticity to place constraints on hadronic form factors. We then derive a basis for functions that obey the constraints imposed on the $B \to B$ elastic form factor F, and show that to 1% accuracy, only two terms in the basis function expansion need be kept. In Sec. 3 we use heavy quark symmetry to relate F to the Isgur-Wise function, which describes the form factors for $\bar{B} \to D^* l\bar{\nu}$ in the infinite quark mass limit. We make a least squares fit to CLEO[2], ARGUS[3], and ALEPH[4] data using $|V_{cb}|$ and our two basis function parameters as variables, and present our results. Reliability of the method is discussed in Sec. 4, implications for $|V_{ub}|$ are discussed in Sec. 5, and concluding remarks are presented in the final section.

2. The Analyticity Constraints

Consider the form factor F, defined by

$$\langle B(p')|V_\mu|B(p)\rangle = F(q^2)(p+p')_\mu, \tag{2.1}$$

where $V_\mu = \bar{b}\gamma_\mu b$, and $q^2 = (p-p')^2$ is the momentum transfer squared. A dispersion relation for the two point function $\langle 0|TV_\mu V_\nu|0\rangle$ connects its perturbative evaluation with a sum over positive-definite terms. This sum includes a contribution $\sim |\langle 0|V_\mu|B\bar{B}\rangle|^2$ which, by crossing, is given by the analytic continuation of F. This procedure leads to a bound[6–8] on a weighted integral of $|F|^2$ over $q^2 > 4M^2$.

A key ingredient in this approach is the transformation that maps the complex q^2 plane onto the unit disc $|z| \leq 1$:

$$\sqrt{1 - \frac{q^2}{4M_B^2}} = \frac{1+z}{1-z} \ . \tag{2.2}$$

In terms of the angular variable $e^{i\theta} \equiv z$, the once-subtracted QCD dispersion relation may be written as

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \, |\phi(e^{i\theta})F(e^{i\theta})|^2 \leq \frac{1}{\pi} \ , \tag{2.3}$$

where the weighing function $\phi(z)$ contains both the Jacobian of the variable transformation and the essential physics of the perturbative QCD calculation[6]:

$$\phi(z) = \frac{1}{16}\sqrt{\frac{5n_f}{6\rho}}(1+z)^2\sqrt{1-z}. \tag{2.4}$$

Here n_f is the number of light flavors for which $SU(n_f)$ flavor symmetry is valid; we take $n_f = 2$. Perturbative corrections to the dispersion relation are incorporated in ρ, which has been computed[9,10] to $\mathcal{O}(\alpha_s)$, $\rho = 1 + 0.73\alpha_s(m_b) \approx 1.20$.

In terms of the inner product defined by

$$(f,g) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \, f^*(\theta)g(\theta) \ , \tag{2.5}$$

and the function

$$\Psi(\theta) = \phi(e^{i\theta})F(e^{i\theta}) \ , \tag{2.6}$$

the dispersion relation Eq. (2.3) reads

$$(\Psi, \Psi) \leq \frac{1}{\pi} \ . \tag{2.7}$$

2

Physically, poles of F inside the unit disc originate from resonances below threshold and cannot be ignored[7]; for the B system these are the resonances $\Upsilon_{1,2,3}$. A simple but effective trick[8] eliminates the poles with no reference to the size of their residues but rather only their positions (*i.e.*, masses). Define the function

$$P(z) = \frac{(z - z_1)(z - z_2)(z - z_3)}{(1 - \bar{z}_1 z)(1 - \bar{z}_2 z)(1 - \bar{z}_3 z)}, \tag{2.8}$$

where the z_i correspond to the values $q^2 = M_{\Upsilon_i}^2$, and the constant

$$T = -P(0)\phi(0)F(0). \tag{2.9}$$

By b-number conservation, $F(0) = 1$. Since $P(z)$ has modulus one on the unit circle, the function

$$f_0(z) = \frac{1}{z}[\phi(z)P(z)F(z) + T] \tag{2.10}$$

is analytic on the unit disk and obeys

$$(f_0, f_0) \leq \frac{1}{\pi} - |T|^2 \equiv I. \tag{2.11}$$

Any function $g(z)$ that is analytic in the unit disc and obeys Eq. (2.11) may be expanded in an orthonormal basis

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.12}$$

where $a_n = (z^n, g)$. The QCD form factor may therefore be written as

$$F(z) = \frac{1}{P(z)\phi(z)} \left[\sum_{n=0}^{\infty} a_n z^{n+1} - T \right], \tag{2.13}$$

with

$$\sum_{n=0}^{\infty} |a_n|^2 \leq \frac{1}{\pi} - |T|^2. \tag{2.14}$$

In the next section we will use heavy quark symmetries to relate F to the form factors for $\bar{B} \to D^* l \bar{\nu}$, where the physical kinematic range is $0 < z < 0.056$. Thus, retaining only a_0 and a_1 induces a maximum relative error of $\sqrt{I}(0.056)^3 / P(0)\phi(0) \approx 0.009$.

3. Extraction of $|V_{cb}|$

3.1. Heavy Quark Symmetry Relations

In the infinite b and c quark mass limit all the form factors for $\bar{B} \to Dl\bar{\nu}$ and $\bar{B} \to D^*l\bar{\nu}$ are given by one universal "Isgur-Wise" function. This allows us to apply the constraint on F to the particular combination of form factors actually measured, rather than deriving constraints for each form factor separately. The Isgur-Wise function is related to the form factor F by $F(\omega) = \eta_B \xi(\omega)$, where $\omega \equiv v \cdot v'$, and the short-distance matching correction η_B is unity at threshold by the Ademollo-Gatto theorem. To compare with data, we need the analogous short-distance matching and running correction[11] for $\bar{B} \to D^*$ form factors. For example, for the vector current form factor g one may write $g = \frac{\eta_D}{\eta_B}F$. This relation generally holds to order $1/M$, but at threshold it holds to order $1/M^2$[12]. We treat η_D/η_B as approximately constant and equal to 0.985. The errors from this approximation should be no larger than the neglected $1/M$ corrections.

3.2. Maximum Likelihood Fit

Once the essential physics of QCD is incorporated into the calculation via Eqs. (2.13) and (2.14), the maximum likelihood fit is simply an ordinary chi-squared minimization with parameters $|V_{cb}|$ and the basis coefficients $\{a_n\}$. As mentioned previously, the smallness of z in the physical range allows us to ignore all coefficients except a_0 and a_1. We normalize input data to a B lifetime[13] of $\tau_B = 1.61$ ps.

$|V_{cb}| \cdot 10^3$	a_0	a_1	Expt.
$35.7^{+4.2}_{-2.8}$	$-0.01^{+0.02}_{-0.06}$	$-0.55^{+0.9}_{-0.0}$	CLEO[2]
$47.6^{+7.9}_{-11.2}$	$-0.11^{+0.10}_{-0.02}$	$0.55^{+0.0}_{-1.1}$	ARGUS[3]
$38.0^{+5.2}_{-4.8}$	$0.03^{+0.04}_{-0.05}$	$-0.55^{+0.5}_{-0.0}$	ALEPH[4]

Table 1 *Fit values for $|V_{cb}|$, a_0, and a_1 from the various experiments.*

Table 1 shows the central values and 68% confidence levels for $|V_{cb}|$, a_0, and a_1 from the various experiments. The saturation of its QCD bound by a_1 is not significant because its variance is large, which arises because the contribution of a_1 is suppressed by an extra power of z.
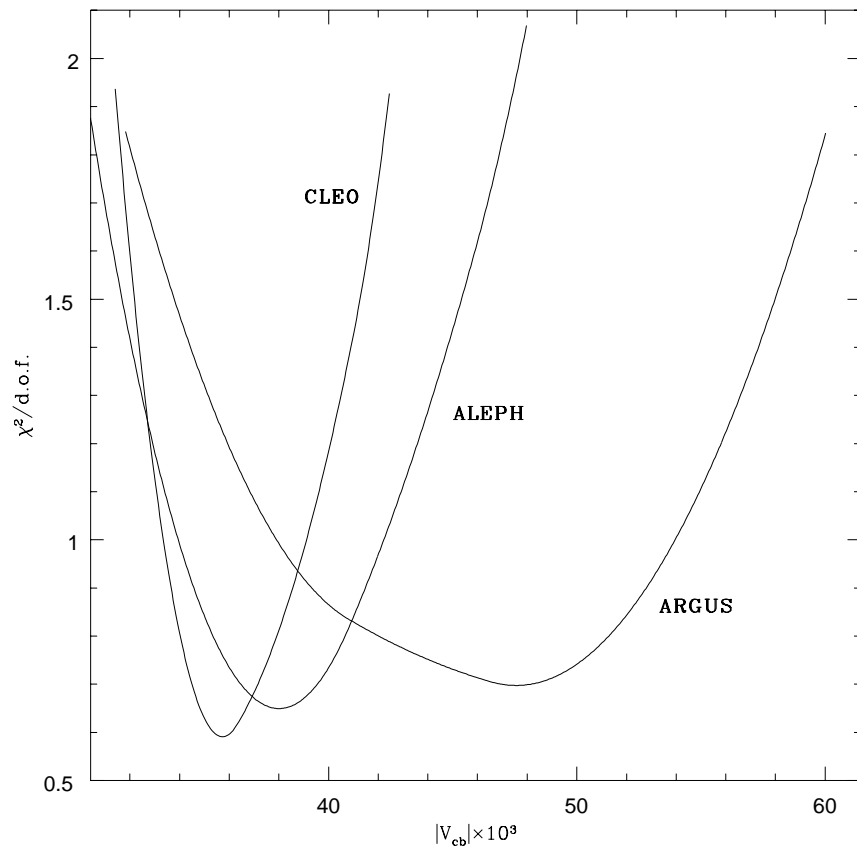
Figure 1 χ^2 per degree of freedom resulting from a least squares fit in $|V_{cb}|$, and a_0 and a_1 from Eq. (2.13) to CLEO, ALEPH, and ARGUS data, plotted against $|V_{cb}|$.

Plots of χ^2 per degree of freedom versus $|V_{cb}|$ for each of the experiments are shown in Fig. 1. The minimum χ^2 is consistently low, remarkable agreement for a first-principles parametrization.

Figure 2 shows the product of the best fit form factors with $|V_{cb}|$, superimposed with experimental data. At 90% confidence level, a_0 and a_1 are consistent with zero, suggesting the dispersion relation may be saturated entirely by higher states.

The errors on $|V_{cb}|$ in Table 1 are statistical only; the treatment of systematic errors depends both on our parametrization and a detailed understanding of the experiment. The error implicit in the variation over choices of parametrization, however, is absent. For the ARGUS experiment, varying over four possible parametrizations induced a spread of 0.012 in $|V_{cb}|$, and was the major impediment in using heavy quark symmetry to obtain a model-independent extraction. Even the experiment with highest statistics, CLEO, remains sensitive to the choice of extraction. For a linear fit, they find $|V_{cb}| \cdot 10^3 = 35.1^{+1.9}_{-1.9}$,

while for a quadratic fit, they find $|V_{cb}| \cdot 10^3 = 35.3^{+3.0}_{-3.2}$. Presumably a higher-order fit would yield even larger variances. Fortunately, the basis function approach does not yield statistical errors indicative of such a higher-order fit.
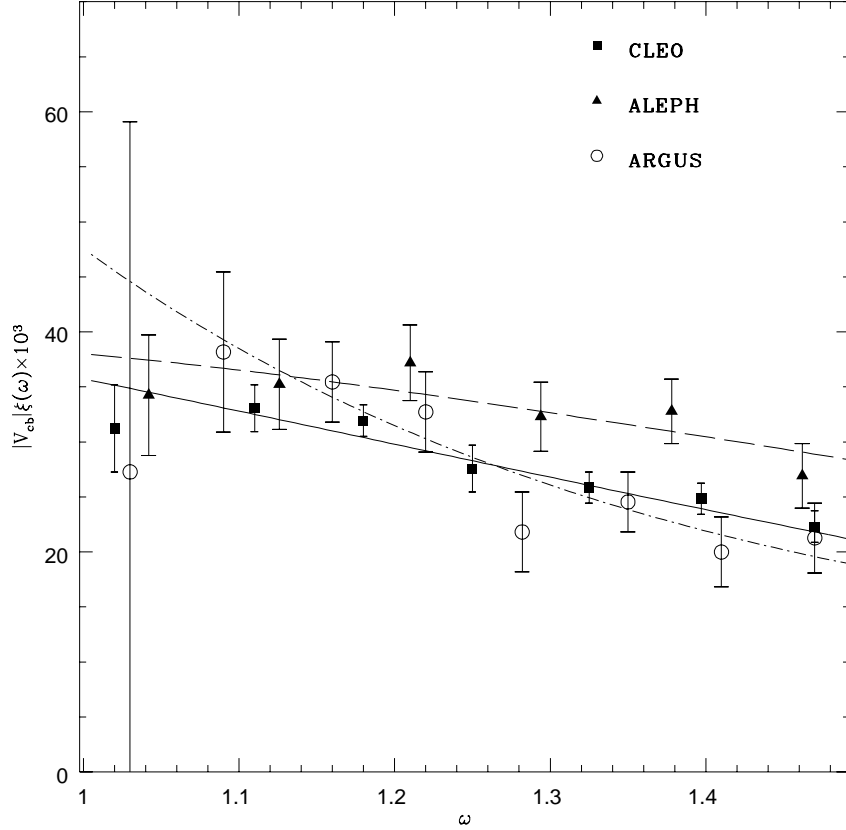

Figure 2 *Best fit values for the product of $|V_{cb}|$ with the Isgur-Wise function for CLEO (solid line), ARGUS (dot-dashed line), and ALEPH (dashed line) data. The data for each experiment, adjusted for the B lifetime and zero-recoil normalization used in the text, is superimposed.*

4. Discussion of the Method

4.1. Reliability of the Parametrization

The parametrization Eq. (2.13) was used to make a three-parameter $(a_0, a_1$ and $|V_{cb}|)$ maximum likelihood fit to the data. We can list four types of correction to this equation.

First, perturbative $\mathcal{O}(\alpha_s^2(m_b))$ corrections to the dispersion relation (2.11) arise as z-independent renormalizations of the function ϕ, and may be calculated systematically

through higher loop diagrams. The extraction of $|V_{cb}|$ is rather insensitive to such corrections: Altering by hand the perturbative computation by $\pm 10\%$ changes the central value of $|V_{cb}|$ by less than $\pm 0.3\%$.

Second, non-perturbative corrections to the dispersion relation may be analyzed via an operator product expansion. The first correction takes the form of a gluon condensate, estimated to be[10,14] $\frac{\alpha_s(M_B)}{M_B^4 \pi} < G_{\mu\nu} G^{\mu\nu} > \approx 10^{-5}$, which is completely negligible.

Third, the truncation of Eq. (2.13) at finite n introduces an error proportional to $\frac{1}{P(0)\phi(0)} \sum_{i=n+1}^{\infty} a_i z^{i+1}$ and additionally suppressed by the coefficients a_n themselves, whose sum is bounded: $\sum_{n=0}^{\infty} |a_n|^2 \leq I \approx 0.31$. Again, for the case at hand, the first two terms are sufficient to describe any form factor allowed by the dispersion relations to within 1%.

Finally, the application of the $B \to B$ dispersion relation to $\bar{B} \to D^* l \bar{\nu}$ decays relies on heavy quark symmetry, which is computationally convenient but unnecessary. The weighing function $\phi(z)$ appropriate to $\bar{B} \to D l \bar{\nu}$ may be readily deduced from an analogous computation for $\bar{B} \to \pi l \bar{\nu}$[15]. Eq. (2.13) then holds with the function $P(z)$ altered to reflect B_c poles below threshold, the positions of which have been calculated in the context of a nonrelativistic potential model[16]. It should also be possible to derive analogous dispersion relations for each of the $\bar{B} \to D^* l \bar{\nu}$ form factors, again using no assumptions about heavy quark symmetry. This should allow the extraction of $|V_{cb}|$ from both $\bar{B} \to D^* l \bar{\nu}$ and $\bar{B} \to D l \bar{\nu}$ decays, given only the normalization of the form factors at zero recoil.

Alternatively, one may want to use heavy quark symmetry to relate the $\bar{B} \to D^* l \bar{\nu}$ and $\bar{B} \to D l \bar{\nu}$ form factors and construct a QCD constraint on only one of them. Such heavy quark symmetry relations are spoiled only by spin-symmetry violating corrections, which are expected to be smaller than the flavor-symmetry violating corrections inherent in the method of Sec. 3.

All of the errors described above are either extremely small, or amenable to systematic reduction. We see no theoretical obstacle to predicting the $\bar{B} \to D$ and $\bar{B} \to D^*$ form factors to 1% accuracy, given the normalization at threshold and sufficiently precise measurements to fix two parameters. Such a prediction would not only test our understanding of QCD at an unprecedented level; it would present a precision probe of non-standard model physics in a hadronic arena.

7

4.2. Reliability of the Extraction

One type of heavy quark symmetry correction arises from the application of the $B \to B$ dispersion relation to $\bar{B} \to D^* l \bar{\nu}$. We estimate such corrections by making a 20% change in the ranges of a_0 and a_1, resulting in a 2% shift in the central value of $|V_{cb}|$.

Another heavy quark correction arises at $\mathcal{O}(\frac{1}{M^2})$ in the normalization of the Isgur-Wise function at threshold. The normalization of the form factor $g(\omega = 1)$ has been estimated to be $g(1) = 0.96$[17], $g(1) = 0.89$[18], and $g(1) = 0.93$[19]. We have included a QCD correction of 0.985, so to good approximation, this simply rescales the values of $|V_{cb}|$ in Table 1 by $\frac{0.985}{g(1)}$.

There are other errors in our extraction that are not purely theoretical. The most pressing of these involve the binning of the measured rate against ω, smearing of ω introduced by boosting from the lab to the center of mass frame, and correlation of errors. Randomly varying input values of ω in our least squares fit of the CLEO data by ± 0.05 changes the central value of $|V_{cb}|$ by less than 1%. A more thorough extraction can be done by the experimental groups themselves, using our basis function expansion in their maximum likelihood programs.

5. Implications for $|V_{ub}|$

The parametrization of form factors in terms of our basis functions applies to other heavy hadron decays, including $\bar{B} \to \pi l \bar{\nu}$. In this case the range of the kinematic variable is larger, $0 < z < 0.5$, so more coefficients a_n are needed for comparable accuracy. We expect six to eight a_n will be necessary for accuracy of a few percent over the entire kinematic range, depending on the form of the actual data.

A least squares fit will only produce best values of the products $|V_{ub}|\xi(1)$ and $|V_{ub}|a_n$, destroying hopes of a model-independent extraction of $|V_{ub}|$. However, small values of $|V_{ub}|$ tend to wash out the nontrivial z dependence, while the $a_n's$ cannot compensate because they are bounded from above, so the extraction of a model-independent lower bound on $|V_{ub}|$ should be possible.

To obtain an upper bound on $|V_{ub}|$ will require as input the overall normalization of the relevant form factor. Practically speaking, this means using a model or lattice simulation. Any candidate model must predict a form factor that is consistent with the basis functions, in the domain of validity of the model. This is a severe test to pass[15], and should serve as an effective discriminator for models.

6.   Conclusions

The extraction of the CKM mixing parameter $|V_{cb}|$ involves several types of uncertainties. Typically, these uncertainties are classified as

$$|V_{cb}| = V \pm \{stat\} \pm \{syst\} \pm \{life\} \pm \{norm\} \pm \{param\} \qquad (6.1)$$

where *stat* and *syst* refer to statistical and systematic experimental uncertainties, *life* refers to uncertainties in the B lifetime, *norm* refers to uncertainty in the value of the form factor at threshold, and *param* refers to uncertainty in the extrapolation of the measured differential rate to threshold.

Not only the central value, but also the statistical uncertainty depends on the parametrization. For example, linear fits to CLEO data yield substantially smaller statistical uncertainties than quadratic fits. Typically, quoted values correspond to the parametrization yielding the smallest statistical uncertainty, in effect throwing some statistical uncertainty into the parametrization uncertainty, which remains implicit. Clearly, this does not improve the accuracy with which we know $|V_{cb}|$.

In this paper, we have essentially eliminated the uncertainty in the choice of parametrization. This was accomplished in four stages. First, we used QCD dispersion relations to constrain the $B \rightarrow B$ elastic form factor. Second, we derived a set of parametrized basis functions which automatically satisfies the dispersion relation constraint, and expressed the $B \rightarrow B$ form factor in terms of this basis. This expression involved an infinite number of parameters a_i bounded by $\sum_{n=0}^{\infty} |a_n|^2 \leq I$. Third, we used heavy quark symmetry to relate the $B \rightarrow B$ form factor to $\bar{B} \rightarrow D^* l \bar{\nu}$ form factors and fixed the normalization at threshold. Over the entire kinematic range relevant to $\bar{B} \rightarrow D^* l \bar{\nu}$, we showed that neglecting all but the first two parameters a_0, a_1 resulted in at most a 1% deviation in the predicted form factor. Finally, we made a least squares fit of the differential $\bar{B} \rightarrow D^* l \bar{\nu}$ rate to $|V_{cb}|$, a_0, and a_1.

The results of this fit improve on all previous extractions in one important way: The uncertainty due to the choice of parametrization is under control, and of order 1%. Our statistical errors are larger than many quoted values. This does not reflect an inferiority of our method, but rather quantifies uncertainties that were previously left implicit. An averaged value from CLEO, ARGUS, and ALEPH data is

$$|V_{cb}| = 0.037^{+0.003}_{-0.002} \, (stat). \qquad (6.2)$$

9

An estimation of systematic uncertainties requires a detailed knowledge of the experiments.

The basis function parametrization described here allows generalization to $\bar{B} \to \pi l \bar{\nu}$ as well. In this case, we expect to be able to extract a lower bound on $|V_{ub}|$. In addition, precision tests of QCD-predicted form factors are now possible; these should be useful as checks of QCD models and lattice simulations. As experiments improve, they may even be used as probes of new physics. Applications and generalizations of the methods described here look promising.

References

[1] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989) and B **237**, 527 (1990) ;
E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990) ;
M. B. Voloshin and M. A. Shifman, Yad. Fiz. **47**, 801 (1988) [Sov. J. Nucl. Phys. **47**, 511 (1988)].

[2] B. Barish *et al.* (CLEO Collaboration), Phys. Rev. D **51**, 1014 (1995).

[3] H. Albrecht *et al.* (ARGUS Collaboration), Z. Phys. C **57**, 533 (1993).

[4] I. Scott (ALEPH Collaboration), to appear in: Proceedings of the 27th International Conference on High Energy Physics, Glasgow, Scotland, July 1994.

[5] N. N. Meiman, Sov. Phys. JETP **17**, 830 (1963) ;
S. Okubo and I. Fushih, Phys. Rev. D **4**, 2020 (1971) ;
V. Singh and A. K. Raina, Fortschritte der Physik **27**, 561 (1979) ;
C. Bourrely, B. Machet, and E. de Rafael, Nucl. Phys. **B189**, 157 (1981).

[6] E. de Rafael and J. Taron, Phys. Lett. B **282**, 215 (1992).

[7] E. Carlson, J. Milana, N. Isgur, T. Mannel, and W. Roberts, Phys. Lett. B **299**, 133 (1993);
A. Falk, M. Luke, and M. Wise, Phys. Lett. B **299**, 123 (1993) ;
B. Grinstein and P. Mende, Phys. Lett. B **299**, 127 (1993) ;
J. Körner and D. Pirjol, Phys. Lett. B **301**, 257 (1993).

[8] E. de Rafael and J. Taron, Phys. Rev. D **50**, 373 (1994).

[9] L. J. Reinders, H. R. Rubinstein, and S. Yazaki, Nucl. Phys. **B186**, 109 (1981).

[10] M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, Phys. Lett. B **77**, 80 (1978) ;
M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).

[11] A. F. Falk, et al, *Nucl. Phys.* **B343** (1990) 1;
A. F. Falk and B. Grinstein, *Phys. Lett.* **B247** (1990) 406;
A. F. Falk and B. Grinstein, *Phys. Lett.* **B249** (1990) 314;
M. Neubert, SLAC Report No. SLAC-PUB-6263.

[12] M. E. Luke, Phys. Lett. B **252**, 447 (1990).

[13] W. Venus, in: Lepton and Photon Interactions, XVI International Symposium, eds. Persis Drell and David Rubin, AIP Press, New York (1994).

[14] A. I. Vainshtein, V. I. Zakharov, and M. A. Shifman, JETP Lett. **27**, 55 (1978).

[15] C. G. Boyd, B. Grinstein, and R. F. Lebed, U. C. San Diego Report No. UCSD/PTH-94/27 [hep-ph/9412324] (unpublished).

[16] E. J. Eichten and C. Quigg, *Phys. Rev.* **D49** (1994) 5845;
M. Baker, J. S. Ball, and F. Zachariasen, *Phys. Rev.* **D45** (1992) 910;
R. Roncaglia *et al.*, *Phys. Rev.* **D51** (1995) 1248;
S. Godfrey and N. Isgur, *Phys. Rev.* **D32** (1985) 189.

[17] T. Mannel, Phys. Rev. D **50**, 428 (1994).

[18] M. Shifman, N. Uraltsev, and A. Vainshtein, Univ. of Minn. Report No. TPI-MINN-94/13-T [hep-ph/9405207] (unpublished).

[19] A. F. Falk and M. Neubert, Phys. Rev. D **47**, 2695 and 2982 (1993) ;
M. Neubert, Phys. Lett. B **338**, 84 (1994).